ADDITIVITY OF FREE GENUS OF KNOTS

MAKOTO OZAWA

ABSTRACT. We show that free genus of knots is additive under connected sum.

1. Introduction

Let K be a knot in the 3-sphere S^3. A Seifert surface F for K in S^3 is said to be *free* if the fundamental group $\pi_1(S^3 - F)$ is a free group. We note that all knots bound free Seifert surfaces, e.g. canonical Seifert surfaces constructed by the Seifert's algorithm. We define the *free genus* $g_f(K)$ of K as the minimal genus over all free Seifert surfaces for K ([5]).

For usual genus, Schubert ([9, 2.10 Proposition]) proved that genus of knots is additive under connected sum. In general, the genus of a knot is not equal to the free genus of it. In fact, free genus may have arbitrarily high gaps with genus ([7], [6]).

In this paper, we show the following theorem.

Theorem 1.1. *For two knots K_1, K_2 in S^3, $g_f(K_1) + g_f(K_2) = g_f(K_1 \# K_2)$.*

2. Preliminaries

We can deform a Seifert surface F by an isotopy so that $F \cap N(K) = N(\partial F; F)$. We denote the exterior $cl(S^3 - N(K))$ by $E(K)$, and the exterior $cl(S^3 - N(F))$ or $cl(E(K) - N(F))$ by $E(F)$. We have the following proposition.

Proposition 2.1. ([2, 5.2], [4, IV.15], [8, Lemma 2.2.]) *A Seifert surface F is free if and only if $E(F)$ is a handlebody.*

We have the following inequality.

Proposition 2.2. $g_f(K_1) + g_f(K_2) \geq g_f(K_1 \# K_2)$.

Proof. Let F_i $(i = 1, 2)$ be a free Seifert surface of minimal genus for K_i. We construct a Seifert surface F for $K_1 \# K_2$ as the boundary connected sum of F_1 and F_2 naturally. Then $E(F)$ is obtained by a boundary connected sum of $E(F_1)$ and $E(F_2)$. Therefore the exterior of F is a handlebody, and F is free. Hence we have the desired inequality. □

We can specify the $+$-*side* and $-$-*side* of a Seifert surface F for a knot K by the orientability of F. We say that a compressing disk D for F is a $+$-*compressing disk* (resp. $-$-*compressing disk*) if the collar of its boundary lies on the $+$-side (resp. $-$-side) of F, and F is called $+$-*compressible* (resp. $-$-*compressible*) if F has a $+$-*compressing disk* (resp. $-$-*compressing disk*). A Seifert surface is said to be *weakly reducible* if there exist a $+$-compressing disk D^+ and a $-$-compressing disk D^- for F such that $\partial D^+ \cap \partial D^- = \emptyset$. Otherwise F is *strongly irreducible*. The Seifert surface F is *reducible* if $\partial D^+ = \partial D^-$. Otherwise F is *irreducible*.

Proposition 2.3. *A free Seifert surface of minimal genus is irreducible.*

Proof. Suppose that F is reducible. Then there exist a $+$-compressing disk D^+ and a $-$-compressing disk D^- for F such that $\partial D^+ = \partial D^-$. By a compression of F along D^+ (this is the same to a compression along D^-), we have a new Seifert surface F'. Since $E(F')$ is homeomorphic to a component of the manifold which is obtained by cutting $E(F)$ along $D^+ \cup D^-$, it is a handlebody. Hence F' is free, but it has a lower genus than F. This contradicts the minimality of F. $\square$

For a free Seifert surface F of minimal genus for $K_1 \# K_2$ and a decomposing sphere S for the connected sum of K_1 and K_2, we will show ultimately that S can be deformed by an isotopy so that S intersects F in a single arc, and we have the equality in Theorem 1. To do this, we divide the proof of Theorem 1 into two cases; (1) F is strongly irreducible, (2) F is weakly reducible. The case (1) is treated in the next section and we consider the case (2) in Section 4.

3. Proof of Theorem 1 (strongly irreducible case)

If a free Seifert surface F of minimal genus for $K_1 \# K_2$ is incompressible, then an innermost loop argument shows that a decomposing sphere S for $K_1 \# K_2$ can be deformed by an isotopy so that S intersects F in a single arc, and by Proposition 3, we have the equality in Theorem 1.

So, hereafter we suppose that F is compressible and that in this section, F is strongly irreducible. Without loss of generality, we may assume that there is a $+$-compressing disk for F. Let $\mathcal{D}^+$ be a $+$-compressing disk system for F, and let F' be a surface obtained by compressing F along $\mathcal{D}^+$. We can retake $\mathcal{D}^+$ so that F' is connected since $E(F)$ is a handlebody. Take $\mathcal{D}^+$ to be maximal with respect to above conditions. We deform F' by an isotopy so that $F' \cap F = K$. Put $A = \partial N(K_1 \# K_2) - IntN(F)$, and let H be a closed surface which is obtained by pushing $F \cup A \cup F'$ into the interior of $E(F')$. Let A_0 be a vertical annulus connecting a core of A and a core of the copy of A in H. Then H bounds a handlebody V in $E(F')$ since V is obtained from $E(F)$ by cutting along $\mathcal{D}^+$. The remainder $W = E(F') - IntV$ is a compression body since it is obtained from $N(\partial E(F'); E(F'))$ by adding 1-handles $N(\mathcal{D}^+)$.

Lemma 3.1. *F' is incompressible in S^3.*

Proof. We specify the $\pm$-side of F' endowed from F naturally. Suppose that F' is $+$-compressible, and let E^+ be a $+$-compressible disk for F'. Then we can regard E^+ as a ∂-reducing disk for $E(F')$. By applying our situation to [1, Lemma 1.1.], we may assume that $E^+ \cap \mathcal{D}^+ = \emptyset$. If ∂E^+ separates F', then E^+ cuts off a handlebody from $E(F')$, and there is a non-separating disk in it. So, we may assume that ∂E^+ is non-separating in F'. Then $\mathcal{D}^+ \cup E^+$ is a $+$-compressing disk system satisfying the previous conditions. This contradicts the maximality of $\mathcal{D}^+$.

Next, suppose that F' is $-$-compressible, and let E^- be a $-$-compressing disk for F'. Then we can regard E^- as a ∂-reducing disk for $E(F')$. By applying our situation to [1, Lemma 1.1.], we may assume that $E^- \cap H = E^- \cap F$ is a single loop, and by exchanging $\mathcal{D}^+$ if it is necessary, that E^- does not intersect $\mathcal{D}^+$. But this contradicts the strongly irreducibility of F. $\square$

By Lemma 5, We can deform the decomposing sphere S by an isotopy so that S intersects F' in a single arc. Put $E(S) = S \cap E(F')$. Then $E(S)$ is a ∂-reducing

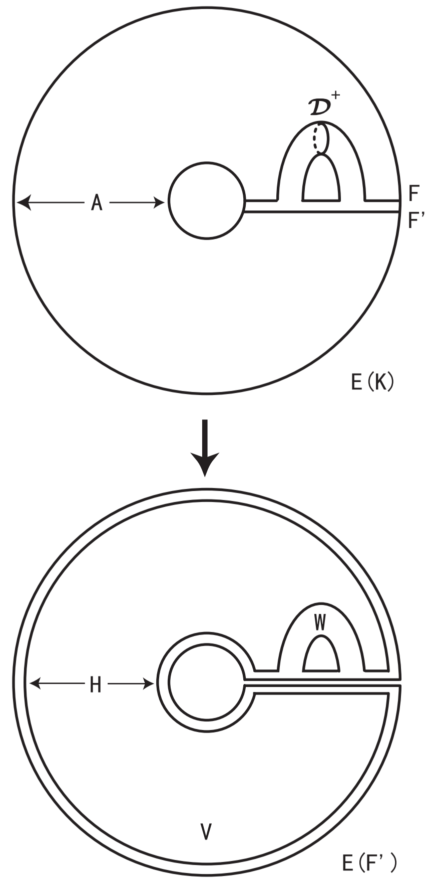

FIGURE 1. Construction of a Heegaard splitting of $E(F')$

disk for $E(F')$. Otherwise, at least one of K_1 or K_2 is trivial, and Theorem 1 clearly holds. By applying our situation to [1, Lemma 1.1.], we may assume that $E(S)$ intersects H in a single loop, $E(S)$ intersects A_0 in two vertical arc, and by exchanging $\mathcal{D}^+$ under the previous conditions if it is necessary, that $E(S)$ does not intersects $\mathcal{D}^+$. Then S intersects F in a single arc, hence we obtain the inequality $g_f(K_1) + g_f(K_2) \leq g_f(K_1 \# K_2)$. This and Proposition 3 complete the Proof of Theorem 1.

4. Proof of Theorem 1 (weakly reducible case)

In this section, we consider the case that F is weakly compressible.

We use the *Hayashi-Shimokawa (HS-) complexity* ([3]). Here we review it. Let H be a closed (possibly disconnected) 2-manifold. Put $w(H) = \{genus(T)|T$ is a component of $H\}$, where this "multi-set" may contain the same ordered pairs redundantly. We order finite multi-sets as follows: arrange the elements of the multi-set in each one in monotonically non-increasing order, then compare the elements lexicographically. We define the HS-complexity $c(H)$ as a multi-set obtained from $w(H)$ by deleting all the 0 elements. We order $c(H)$ in the same way as w.

Let α be a 1-submanifold of H. Then let $\rho(H, \alpha)$ denote the closed 2-manifold obtained by cutting H along α and capping off the resulting two boundary circles with disks.

Since F is weakly reducible, there exist $+$-compressing disk D^+ and $-$-compressing disk D^- for F such that $\partial D^+ \cap \partial D^- = \emptyset$. If $c(\rho(F; \partial D^+ \cup \partial D^-)) = c(\rho(F; \partial D^+))$, say, then ∂D^- bounds a $+$-compressing disk for F. Hence F is reducible, and by Proposition 4, a contradiction.

Therefore there exist non-empty $+$-compressing disks system $\mathcal{D}^+$ and $-$-compressing disk system $\mathcal{D}^-$ for F such that

(1) $\partial\mathcal{D}^+ \cap \partial\mathcal{D}^- = \emptyset$,
(2) $c(\rho(F; \partial\mathcal{D}^+ \cup \partial\mathcal{D}^-)) < c(\rho(F; \partial\mathcal{D}^+))$, $c(\rho(F; \partial\mathcal{D}^-))$,

and with $c(\rho(F; \partial\mathcal{D}^+ \cup \partial\mathcal{D}^-))$ minimal subject to these conditions. Moreover we take $\mathcal{D}^\pm$ so that $|\mathcal{D}^\pm|$ is minimal.

Let $F^\pm$ be a 2-manifold obtained by compressing F along $\mathcal{D}^\pm$, and F' be a 2-manifold obtained by compressing F along $\mathcal{D}^+ \cup \mathcal{D}^-$. We deform F^+ and F^- by an isotopy so that $F^+ \cap F' \cap F^- = K$ and $F^\pm \cap N(K) = N(\partial F^\pm; F^\pm)$. Put $A = \partial N(K_1 \# K_2) - IntN(F)$, and let H be a closed 2-manifold which is obtained by pushing $F^+ \cup A \cup F^-$ into the interior of $E(F')$. Let A_0 be a vertical annulus connecting a core of A and a core of the copy of A in H. Then H bounds the union of handlebodies V in $E(F')$ since V is obtained from $E(F)$ by cutting along $\mathcal{D}^\pm$. The remainder $W = E(F') - IntV$ is a union of compression bodies since it is obtained from $N(\partial E(F'); E(F'))$ by adding 1-handles $N(\mathcal{D}^\pm)$.

Lemma 4.1. *There is no 2-sphere component of H.*

Proof. Suppose that there is a 2-sphere component H_i of H. We may assume that H does not contain A, and there is a copy of some component of $\mathcal{D}^+$ in H. Let $\mathcal{D}^+_s$ be a subsystem of $\mathcal{D}^+$ the union of whose boundaries separates F. If there is no copy of $\mathcal{D}^-$ in H_i, then we delete any one of $\mathcal{D}^+_s$. Then $\mathcal{D}^\pm$ holds the previous conditions, but this contradicts the minimality of $|\mathcal{D}^+|$. If there is a copy of $\mathcal{D}^-$ in H_i, then there is a simple closed curve in H_i which separates $N(\mathcal{D}^+) \cap H_i$ from $N(\mathcal{D}^-) \cap H_i$, and bounds a $+$-compressing disk and $-$-compressing disk for F. Hence F is reducible, but this contradicts Proposition 4.

□

Lemma 4.2. *Each component of F' is incompressible in S^3.*

Proof. We specify the $\pm$-side of $F^\pm$ and F' endowed from F naturally. Suppose without loss of generality that F' is $+$-compressible, and let E^+ be a $+$-compressing disk for F'. Then we can regard E^+ as a ∂-reducing disk for $E(F')$. By applying our situation to [1, Lemma 1.1.], we may assume that E^+ intersects H in a single loop which does not intersect A_0. We deform E^+ by an isotopy so that $E^+ \cap \mathcal{D}^+ = \emptyset$ in S^3. We take a complete meridian disk system $\mathcal{C}$ of W which includes $\mathcal{D}^+$ and does not intersect E^+. Put $\mathcal{C}^- = \mathcal{C} - \mathcal{D}^+$. Then we have $c(\rho(F; \partial E^+ \cup \partial\mathcal{D}^+ \cup \partial\mathcal{C}^-))$ $< c(\rho(F; \partial\mathcal{D}^+ \cup \partial\mathcal{C}^-))$ since ∂E^+ is essential in F'. Suppose that $c(\rho(F; \partial E^+ \cup \partial\mathcal{D}^+ \cup \partial\mathcal{C}^-)) = c(\rho(F; \partial E^+ \cup \partial\mathcal{D}^+))$. Then each component of $\partial\mathcal{D}^-$ bounds both $+$-compressing disk and $-$-compressing disk for F. Hence F is reducible, but this contradicts Proposition 2.3. Similarly, if $c(\rho(F; \partial E^+ \cup \partial\mathcal{D}^+ \cup \partial\mathcal{C}^-)) = c(\rho(F; \partial\mathcal{C}^-))$, then we are done. Hence we obtain a $\pm$-compressing disk system $E^+ \cup \mathcal{D}^+$, $\mathcal{C}^-$ for

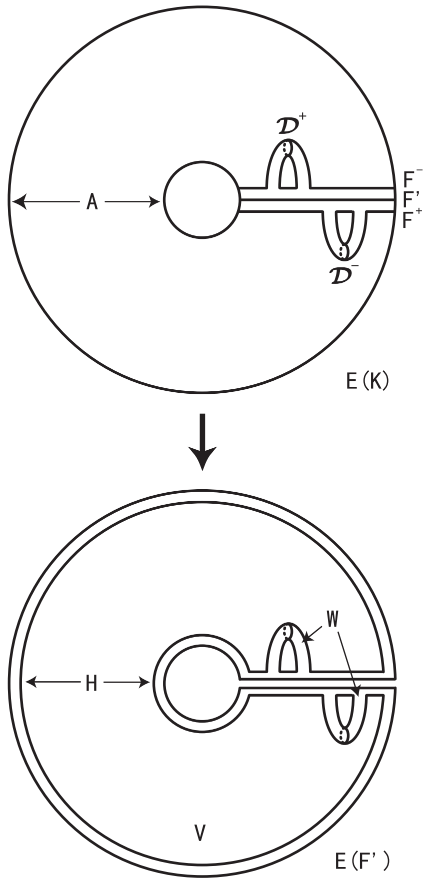

FIGURE 2. Construction of a Heegaard splitting of $E(F')$

F which satisfies the conditions (1), (2) and have more minimal complexity than $\mathcal{D}^+ \cup \mathcal{D}^-$. This contradicts the property of $\mathcal{D}^+ \cup \mathcal{D}^-$.

□

By Lemma 7, we can deform the decomposing sphere S by an isotopy so that S intersects F' in a single arc. Put $E(S) = S \cap E(F')$. Then $E(S)$ is a ∂-reducing disk for $E(F')$. Otherwise, at least one of K_1 and K_2 is trivial, and Theorem 1 clearly holds. Let V_0 and W_0 be components of V and W respectively, where V_0 contains A and W_0 is the next handlebody to V_0. Put $H_0 = V_0 \cap W_0$. Then H_0 gives a Heegaard splitting of $V_0 \cup W_0$. By Lemma 7, we can deform $E(S)$ by an isotopy so that $E(S)$ is contained in $V_0 \cup W_0$. By applying this situation to [1, Lemma 1.1] or [3, Theorem 1.3], we may assume that $E(S)$ intersects H_0 in a single loop without moving $\partial E(S)$. Moreover, there exist a complete meridian disk system $\mathcal{E}_0$ of V_0 such that $\mathcal{E}_0 \cap E(S) = \emptyset$ and $\mathcal{E}_0 \cap A_0 = \emptyset$. Thus S intersects F in a single arc, hence we have the conclusion.

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Department of Mathematics, School of Education, Waseda University, 1-6-1 Nishi-waseda, Shinjuku-ku, Tokyo 169-8050, Japan

E-mail address: ozawa@mn.waseda.ac.jp